J. Math. Anal. Appl. 372 (2010) 369–376

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Lie derivable maps on $B(X)$ ☆

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ARTICLE INFO

Article history:
Received 6 June 2009
Available online 21 July 2010
Submitted by R. Curto

Keywords:
Lie derivation
Lie derivable map
Additivity
Rank one operator

ABSTRACT

Let X be a Banach space of dimension greater than 1. We prove that if a map $\delta : B(X) \to B(X)$ satisfies

$$\delta\big([A, B]\big) = \big[\delta(A), B\big] + \big[A, \delta(B)\big]$$

for any $A, B \in B(X)$, then $\delta = D + \tau$, where D is an additive derivation of $B(X)$ and the map $\tau : B(X) \to \mathbb{F}I$ vanishes at commutators $[A, B]$.

1. Introduction and statement of result

Let X be a Banach space over $\mathbb{F}$, where $\mathbb{F}$ is the real number field $\mathbb{R}$ or the complex field $\mathbb{C}$. By X^* and $B(X)$, we denote the topological dual space of X and the algebra of all linear bounded operators on X, respectively. If $x \in X$ and $f \in X^*$, the rank one operator $x \otimes f$ is defined by $y \mapsto f(y)x$ for $y \in X$.

Recall that an additive map δ from $B(X)$ into itself is called an additive derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in B(X)$ and an additive Lie derivation if $\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$ for all $A, B \in B(X)$. Here $[A, B] = AB - BA$ is the usual Lie product. From the classical result in [15], we know that every additive Lie derivation on $B(X)$ can be expressed as the sum of an additive derivation and an additive map with image in the center vanishing at commutators. In this note, we obtain basically the same structure withdrawing the additivity condition. More precisely, our main result reads as follows.

Theorem 1.1. *Let X be a Banach space of dimension > 1. Suppose that the map $\delta : B(X) \to B(X)$ satisfies*

$$\delta\big([A, B]\big) = \big[\delta(A), B\big] + \big[A, \delta(B)\big] \quad \text{for all } A, B \in B(X). \tag{1.1}$$

Then $\delta = D + \tau$, where D is an additive derivation, τ is a map from $B(X)$ into $\mathbb{F}I$ satisfying $\tau([A, B]) = 0$ for all $A, B \in B(X)$.

We remark that there has been a great interest in the study of Lie derivations in recent years. For example, the structure of Lie derivations has been investigated for prime rings in [5,15,23,24,2], for C*-algebras and for more general semisimple Banach algebras in [9,17,18,1,25], for triangular algebras in [6,3,4,14]. On the other hand, our investigation is also motivated by other related topic. For example, papers [10,8,21,22,12,16] studied when a multiplicative map is additive, the paper [7] studied when a derivable map is additive, papers [11,19,20,13] studied when a Jordan multiplicative map is additive, the paper [26] studied the structure of Lie multiplicative maps.

☆ The research was supported by NNSFC (No. 10771154) and PNSFJ (No. BK2007049).
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.07.002

2. Proof

In this section we shall prove our main theorem. For the convenience of citation and clarity of exposition, we shall organize the proof in a series of lemmas. We begin with the trivial one.

Lemma 2.1. $\delta(0) = 0$.

Proof. Indeed, $\delta(0) = \delta([0,0]) = [\delta(0), 0] + [0, \delta(0)] = 0$. $\square$

In what follows, take $x_0 \in X$, $f_0 \in X^*$ satisfying $f_0(x_0) = 1$. Let $P_1 = x_0 \otimes f_0$ and $P_2 = I - P_1$. Let $\mathscr{A}_{ij} = P_i B(X) P_j$, $1 \leqslant i, j \leqslant 2$. Then $B(X) = \sum_{i,j=1}^{2} \mathscr{A}_{ij}$.

Lemma 2.2. $P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2 \in \mathbb{F}I$.

Proof. Let $x \in X$, $f \in X^*$. Then

$$\begin{aligned} \delta\big(P_1x \otimes P_2^* f\big) &= \delta\big(\big[P_1, P_1x \otimes P_2^* f\big]\big) \\ &= \big[\delta(P_1), P_1x \otimes P_2^* f\big] + \big[P_1, \delta\big(P_1x \otimes P_2^* f\big)\big] \\ &= \delta(P_1)\big(P_1x \otimes P_2^* f\big) - \big(P_1x \otimes P_2^* f\big)\delta(P_1) + P_1\delta\big(P_1x \otimes P_2^* f\big) - \delta\big(P_1x \otimes P_2^* f\big)P_1. \end{aligned}$$

Multiplying this equation by P_1 from the left and by P_2 from the right, we get, for all $x \in X$ and $f \in X^*$, that

$$P_1\delta(P_1)P_1(x \otimes f)P_2 = P_1(x \otimes f)P_2\delta(P_1)P_2,$$

from which we see that $P_1\delta(P_1)P_1 = \lambda P_1$ for some $\lambda \in \mathbb{F}$. Hence $P_2\delta(P_1)P_2 = \lambda P_2$. Thus, $P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2 = \lambda I$, proving the lemma. $\square$

Now let $T = P_1\delta(P_1)P_2 - P_2\delta(P_1)P_1$. For $A \in B(X)$, define $\Delta(A) = \delta(A) - (AT - TA)$. Then $\Delta(P_1) \in \mathbb{F}I$ and Δ also satisfies Eq. (1.1). The excellent features of the map Δ will be shown in Lemmas 2.5 and 2.6 which say that Δ leaves each $\mathscr{A}_{ij}$ invariant up to scalar summands.

Lemma 2.3. *For $A \in B(X)$, we have $\Delta(P_1AP_2 + P_2AP_1) = P_1\Delta(A)P_2 + P_2\Delta(A)P_1$.*

Proof. Since

$$\big[P_1, [P_1, A]\big] = P_1A - 2P_1AP_1 + AP_1 = P_1AP_2 + P_2AP_1,$$

it follows that

$$\begin{aligned} \Delta(P_1AP_2 + P_2AP_1) &= \Delta\big(\big[P_1, [P_1, A]\big]\big) = \big[P_1, \big[P_1, \Delta(A)\big]\big] \\ &= P_1\Delta(A)P_2 + P_2\Delta(A)P_1, \end{aligned}$$

proving the lemma. $\square$

Lemma 2.4. $\Delta(P_2) \in \mathbb{F}I$.

Proof. Using an argument similar to that in the proof of Lemma 2.2, we arrive at

$$P_1\Delta(P_2)P_1 + P_2\Delta(P_2)P_2 \in \mathbb{F}I.$$

On the other hand, by Lemma 2.3,

$$P_1\Delta(P_2)P_2 + P_2\Delta(P_2)P_1 = \Delta(P_1P_2P_2 + P_2P_2P_1) = 0.$$

Consequently, $\Delta(P_2) = P_1\Delta(P_2)P_1 + P_2\Delta(P_2)P_2 \in \mathbb{F}I$. $\square$

Lemma 2.5. $\Delta(\mathscr{A}_{ij}) \subseteq \mathscr{A}_{ij}$, $1 \leqslant i \neq j \leqslant 2$.

Proof. For $A_{12} \in \mathscr{A}_{12}$, we have

$$\Delta(A_{12}) = \Delta\big([P_1, A_{12}]\big) = \big[P_1, \Delta(A_{12})\big] = P_1\Delta(A_{12})P_2 - P_2\Delta(A_{12})P_1,$$

from which we see that

$$P_1\Delta(A_{12})P_1 = P_2\Delta(A_{12})P_2 = P_2\Delta(A_{12})P_1 = 0.$$

So, $\Delta(A_{12}) = P_1\Delta(A_{12})P_2 \in \mathscr{A}_{12}$ for each $A_{12} \in \mathscr{A}_{12}$. This implies that $\Delta(\mathscr{A}_{12}) \subseteq \mathscr{A}_{12}$.

Similarly, $\Delta(A_{21}) = P_2\Delta(A_{21})P_1 \in \mathscr{A}_{21}$ for any $A_{21} \in \mathscr{A}_{21}$, and therefore $\Delta(\mathscr{A}_{21}) \subseteq \mathscr{A}_{21}$. $\square$

Lemma 2.6. *There is a functional $f_i : \mathscr{A}_{ii} \to \mathbb{F}$ such that $\Delta(A_{ii}) - f_i(A_{ii})I \in \mathscr{A}_{ii}$ for all $A_{ii} \in \mathscr{A}_{ii}$, $1 \leqslant i \leqslant 2$.*

Proof. For $A_{ii} \in \mathscr{A}_{ii}$, by Lemma 2.3, we have that

$$P_1\Delta(A_{ii})P_2 + P_2\Delta(A_{ii})P_1 = \Delta(P_1A_{ii}P_2 + P_2A_{ii}P_1) = \Delta(0) = 0.$$

Therefore, we may suppose that $\Delta(A_{11}) = X_{11} + X_{22}$ and $\Delta(A_{22}) = Y_{11} + Y_{22}$. Here $X_{ii}, Y_{ii} \in \mathscr{A}_{ii}$, $i = 1, 2$. Thus

$$0 = \Delta\big([A_{11}, A_{22}]\big) = \big[\Delta(A_{11}), A_{22}\big] + \big[A_{11}, \Delta(A_{22})\big] = [X_{22}, A_{22}] + [A_{11}, Y_{11}].$$

Consequently, $[X_{22}, A_{22}] = 0$ for all $A_{22} \in \mathscr{A}_{22}$; $[A_{11}, Y_{11}] = 0$ for all $A_{11} \in \mathscr{A}_{11}$. Therefore, there exist scalars $f_1(A_{11})$ and $f_2(A_{22})$ such that $X_{22} = f_1(A_{11})P_1$ and $Y_{11} = f_2(A_{22})P_2$. So $\Delta(A_{11}) - f_1(A_{11})I \in \mathscr{A}_{11}$ and $\Delta(A_{22}) - f_2(A_{22})I \in \mathscr{A}_{22}$. $\square$

The next goal is to show that Δ is additive on $\mathscr{A}_{12}$ and $\mathscr{A}_{21}$.

Lemma 2.7. *Let $A_{ii} \in \mathscr{A}_{ii}$ and $A_{ij} \in \mathscr{A}_{ij}$, $1 \leqslant i \neq j \leqslant 2$. Then $\Delta(A_{ii} + A_{ij}) - \Delta(A_{ii}) - \Delta(A_{ij}) \in \mathbb{F}I$.*

Proof. We shall prove the lemma only for the case $i = 1$ and $j = 2$; the proof for the other cases is similar.

Let $A_{11} \in \mathscr{A}_{11}$, $A_{12} \in \mathscr{A}_{12}$. Then for $B_{12} \in \mathscr{A}_{12}$, by Lemmas 2.5 and 2.6 we have

$$\begin{aligned}\Delta\big([A_{11} + A_{12}, B_{12}]\big) &= \big[\Delta(A_{11} + A_{12}), B_{12}\big] + \big[A_{11} + A_{12}, \Delta(B_{12})\big]\\ &= \big[\Delta(A_{11} + A_{12}), B_{12}\big] + \big[A_{11}, \Delta(B_{12})\big],\end{aligned}$$

and

$$\begin{aligned}\Delta\big([A_{11} + A_{12}, B_{12}]\big) = \Delta\big([A_{11}, B_{12}]\big) &= \big[\Delta(A_{11}), B_{12}\big] + \big[A_{11}, \Delta(B_{12})\big]\\ &= \big[\Delta(A_{11}) + \Delta(A_{12}), B_{12}\big] + \big[A_{11}, \Delta(B_{12})\big].\end{aligned}$$

Comparing these two equations, we get

$$\big[\Delta(A_{11} + A_{12}) - \Delta(A_{11}) - \Delta(A_{12}), B_{12}\big] = 0,$$

that is,

$$\big(\Delta(A_{11} + A_{12}) - \Delta(A_{11}) - \Delta(A_{12})\big)B_{12} = B_{12}\big(\Delta(A_{11} + A_{12}) - \Delta(A_{11}) - \Delta(A_{12})\big) \tag{2.1}$$

holds for all $B_{12} \in \mathscr{A}_{12}$. Similarly, since $\Delta([A_{11} + A_{12}, B_{22}]) = \Delta([A_{12}, B_{22}])$,

$$\big(\Delta(A_{11} + A_{12}) - \Delta(A_{11}) - \Delta(A_{12})\big)B_{22} = B_{22}\big(\Delta(A_{11} + A_{12}) - \Delta(A_{11}) - \Delta(A_{12})\big) \tag{2.2}$$

holds for all $B_{22} \in \mathscr{A}_{22}$. Thus, by (2.1) and (2.2), for $S \in B(X)$ there holds

$$\big(\Delta(A_{11} + A_{12}) - \Delta(A_{11}) - \Delta(A_{12})\big)SP_2 = SP_2\big(\Delta(A_{11} + A_{12}) - \Delta(A_{11}) - \Delta(A_{12})\big).$$

In particular, for $x \in X$, $f \in X^*$ we have

$$\big(\Delta(A_{11} + A_{12}) - \Delta(A_{11}) - \Delta(A_{12})\big)x \otimes P_2^*f = x \otimes P_2^*f\big(\Delta(A_{11} + A_{12}) - \Delta(A_{11}) - \Delta(A_{12})\big),$$

from which we see that $\Delta(A_{11} + A_{12}) - \Delta(A_{11}) - \Delta(A_{12}) \in \mathbb{F}I$. $\square$

Lemma 2.8. *Δ is additive on $\mathscr{A}_{12}$ and $\mathscr{A}_{21}$.*

Proof. Let $A_{12}, B_{12} \in \mathscr{A}_{12}$. By Lemma 2.7, we have

$$\begin{aligned}\Delta(A_{12} + B_{12}) &= \Delta\big([P_1 + A_{12}, P_2 + B_{12}]\big)\\ &= \big[\Delta(P_1 + A_{12}), P_2 + B_{12}\big] + \big[P_1 + A_{12}, \Delta(P_2 + B_{12})\big]\\ &= \big[\Delta(P_1) + \Delta(A_{12}), P_2 + B_{12}\big] + \big[P_1 + A_{12}, \Delta(P_2) + \Delta(B_{12})\big]\\ &= \big[\Delta(A_{12}), P_2 + B_{12}\big] + \big[P_1 + A_{12}, \Delta(B_{12})\big]\\ &= \Delta(A_{12}) + \Delta(B_{12}).\end{aligned}$$

So Δ is additive on $\mathscr{A}_{12}$. Similarly, one can verify the additivity of Δ on $\mathcal{A}_{21}$. $\square$

Now, for $A \in B(X)$, define $D(A) = \sum_{i,j=1}^{2} \Delta(P_i A P_j) - (f_1(P_1 A P_1) + f_2(P_2 A P_2))I$. By Lemmas 2.5 and 2.6, we have

Lemma 2.9. *Let $A_{ij} \in \mathscr{A}_{ij}$, $1 \leqslant i, j \leqslant 2$. Then*

(1) $D(A_{ij}) \in \mathscr{A}_{ij}$, $1 \leqslant i, j \leqslant 2$;
(2) $D(A_{12}) = \Delta(A_{12})$ *and* $D(A_{21}) = \Delta(A_{21})$;
(3) $D(A_{11} + A_{12} + A_{21} + A_{22}) = D(A_{11}) + D(A_{12}) + D(A_{21}) + D(A_{22})$.

In the following, we shall show that D is an additive derivation. First, we prove the additivity of D. By Lemma 2.9(3), it is enough to show that D is additive on each row.

The following lemma immediately follows from Lemma 2.8 and Lemma 2.9(2).

Lemma 2.10. *D is additive on $\mathscr{A}_{12}$ and $\mathscr{A}_{21}$.*

Lemma 2.11. *Let $A_{ii} \in \mathscr{A}_{ii}$, $B_{ij} \in \mathscr{A}_{ij}$, $1 \leqslant i \neq j \leqslant 2$. Then*

$$D(A_{ii} B_{ij}) = D(A_{ii}) B_{ij} + A_{ii} D(B_{ij}),$$
$$D(B_{ij} A_{jj}) = D(B_{ij}) A_{jj} + B_{ij} D(A_{jj}).$$

Proof. Since $D(A_{11}) \in \mathscr{A}_{11}$ and $D(B_{12}) \in \mathscr{A}_{12}$, we have

$$\begin{aligned} D(A_{11} B_{12}) &= \Delta\bigl([A_{11}, B_{12}]\bigr) \\ &= \bigl[D(A_{11}), B_{12}\bigr] + \bigl[A_{11}, D(B_{12})\bigr] \\ &= D(A_{11}) B_{12} + A_{11} D(B_{12}). \end{aligned}$$

The other three identities can be similarly proved. $\square$

Lemma 2.12. *D is additive on $\mathscr{A}_{11}$ and $\mathscr{A}_{22}$.*

Proof. Let $A_{11}, B_{11} \in \mathscr{A}_{11}$. For $B_{12} \in \mathscr{A}_{12}$, by Lemma 2.11 we have

$$D\bigl((A_{11} + B_{11}) B_{12}\bigr) = D(A_{11} + B_{11}) B_{12} + (A_{11} + B_{11}) D(B_{12});$$

on the other hand, by Lemmas 2.10 and 2.11 we have

$$\begin{aligned} D\bigl((A_{11} + B_{11}) B_{12}\bigr) &= D(A_{11} B_{12} + B_{11} B_{12}) \\ &= D(A_{11} B_{12}) + D(B_{11} B_{12}) \\ &= D(A_{11}) B_{12} + D(B_{11}) B_{12} + A_{11} D(B_{12}) + B_{11} D(B_{12}). \end{aligned}$$

Comparing these two equations, we get

$$D(A_{11} + B_{11}) B_{12} = \bigl(D(A_{11}) + D(B_{11})\bigr) B_{12}.$$

This is equivalent to

$$\bigl(D(A_{11} + B_{11}) - D(A_{11}) - D(B_{11})\bigr) P_1 B(X) P_2 = 0.$$

Since $B(X)$ is prime, it follows that

$$\bigl(D(A_{11} + B_{11}) - D(A_{11}) - D(B_{11})\bigr) P_1 = 0,$$

and hence $D(A_{11} + B_{11}) = D(A_{11}) + D(B_{11})$ since $D(\mathscr{A}_{11}) \subseteq \mathscr{A}_{11}$.

Similarly, one can prove that D is additive on $\mathscr{A}_{22}$. $\square$

Lemma 2.13. *D is additive.*

Proof. Let $A, B \in B(X)$ and write $A = \sum_{i,j=1}^{2} A_{ij}$, $B = \sum_{i,j=1}^{2} B_{ij}$. By Lemmas 2.9, 2.10 and 2.12, we have

$$\begin{aligned}
D(A+B) &= D\left(\sum_{i,j=1}^{2} A_{ij} + \sum_{i,j=1}^{2} B_{ij}\right) \\
&= \sum_{i,j=1}^{2} D(A_{ij}+B_{ij}) = \sum_{i,j=1}^{2} \big(D(A_{ij}) + D(B_{ij})\big) \\
&= D\left(\sum_{i,j=1}^{2} A_{ij}\right) + D\left(\sum_{i,j=1}^{2} B_{ij}\right) \\
&= D(A) + D(B). \qquad \square
\end{aligned}$$

In the sequel, we shall show that D satisfies the derivation equation, that is, prove $D(AB) = D(A)B + AD(B)$ for all $A = \sum_{i,j=1}^{2} A_{ij}$, $B = \sum_{i,j=1}^{2} B_{ij}$. By the additivity, it is sufficient to show $D(A_{ij}B_{kl}) = D(A_{ij})B_{kl} + A_{ij}D(B_{kl})$ for $1 \leqslant i, j, k, l \leqslant 2$. Since $D(A_{ij}B_{kl}) = 0 = D(A_{ij})B_{kl} + A_{ij}D(B_{kl})$ whenever $j \neq k$, we only consider case $j = k$.

Lemma 2.14. *Let $A_{ii}, B_{ii} \in \mathscr{A}_{ii}$, $i = 1, 2$. Then $D(A_{ii}B_{ii}) = D(A_{ii})B_{ii} + A_{ii}D(B_{ii})$.*

Proof. For $C_{12} \in \mathscr{A}_{12}$, by Lemma 2.11, on the one hand

$$D(A_{11}B_{11}C_{12}) = D(A_{11}B_{11})C_{12} + A_{11}B_{11}D(C_{12}),$$

on the other hand

$$\begin{aligned}
D(A_{11}B_{11}C_{12}) &= D(A_{11})B_{11}C_{12} + A_{11}D(B_{11}C_{12}) \\
&= D(A_{11})B_{11}C_{12} + A_{11}D(B_{11})C_{12} + A_{11}B_{11}D(C_{12}).
\end{aligned}$$

Comparing these two equations, we get $D(A_{11}B_{11})C_{12} = (D(A_{11})B_{11} + A_{11}D(B_{11}))C_{12}$ for all $C_{12} \in \mathscr{A}_{12}$. Hence $D(A_{11}B_{11}) = D(A_{11})B_{11} + A_{11}D(B_{11})$.

Similarly, we can verify that $D(A_{22}B_{22}) = D(A_{22})B_{22} + A_{22}D(B_{22})$. $\square$

Lemma 2.15. *Let $A_{11} \in \mathscr{A}_{11}$, $B_{22} \in \mathscr{A}_{22}$. Then $\Delta(A_{11} + B_{22}) - D(A_{11}) - D(B_{22}) \in \mathbb{F}I$.*

Proof. For $C_{11} \in \mathscr{A}_{11}$, on the one hand

$$\Delta\big([A_{11} + B_{22}, C_{11}]\big) = \big[\Delta(A_{11} + B_{22}), C_{11}\big] + \big[(A_{11} + B_{22}), D(C_{11})\big],$$

on the other hand

$$\begin{aligned}
\Delta\big([A_{11} + B_{22}, C_{11}]\big) &= \Delta\big([A_{11}, C_{11}]\big) \\
&= \big[\Delta(A_{11}), C_{11}\big] + \big[A_{11}, \Delta(C_{11})\big] \\
&= \big[D(A_{11}), C_{11}\big] + \big[A_{11}, D(C_{11})\big] \\
&= \big[D(A_{11}) + D(B_{22}), C_{11}\big] + \big[A_{11} + B_{22}, D(C_{11})\big].
\end{aligned}$$

Comparing these two equations, we get

$$\big[\Delta(A_{11} + B_{22}), C_{11}\big] = \big[D(A_{11}) + D(B_{22}), C_{11}\big],$$

that is

$$\big(\Delta(A_{11} + B_{22}) - D(A_{11}) - D(B_{22})\big)C_{11} = C_{11}\big(\Delta(A_{11} + B_{22}) - D(A_{11}) - D(B_{22})\big).$$

Therefore, there exists a scalar λ such that $(\Delta(A_{11} + B_{22}) - D(A_{11}) - D(B_{22}))P_1 = \lambda P_1$.

Similarly, there exists a scalar μ such that $(\Delta(A_{11} + B_{22}) - D(A_{11}) - D(B_{22}))P_2 = \mu P_2$.

Now it is sufficient to show $\lambda = \mu$. For this, choose a non-zero element $C_{12} \in \mathscr{A}_{12}$. On the one hand we have

$$\Delta\big([A_{11} + B_{22}, C_{12}]\big) = \big[\Delta(A_{11} + B_{22}), C_{12}\big] + \big[A_{11} + B_{22}, D(C_{12})\big],$$

on the other hand, by Lemmas 2.9, 2.13 and 2.11, we have

$$\begin{aligned}
\Delta\big([A_{11} + B_{22}, C_{12}]\big) &= \Delta(A_{11}C_{12} - C_{12}B_{22}) \\
&= D(A_{11}C_{12} - C_{12}B_{22}) \\
&= D(A_{11})C_{12} + A_{11}D(C_{12}) - D(C_{12})B_{22} - C_{12}D(B_{22}) \\
&= \big[D(A_{11}) + D(B_{22}), C_{12}\big] + \big[A_{11} + B_{22}, D(C_{12})\big].
\end{aligned}$$

Comparing these two equations, we get

$$\big[\Delta(A_{11}+B_{22}), C_{12}\big] = \big[D(A_{11})+D(B_{22}), C_{12}\big],$$

which is equivalent to

$$\big(\Delta(A_{11}+B_{22}) - D(A_{11}) - D(B_{22})\big)C_{12} = C_{12}\big(\Delta(A_{11}+B_{22}) - D(A_{11}) - D(B_{22})\big).$$

So $\lambda C_{12} = \mu C_{12}$ and hence $\lambda = \mu$, completing the proof. □

Lemma 2.16. *Let $A_{12} \in \mathscr{A}_{12}$, $B_{21} \in \mathscr{A}_{21}$. Then $D(A_{12}B_{21}) = D(A_{12})B_{21} + A_{12}D(B_{21})$ and $D(B_{21}A_{12}) = D(B_{21})A_{12} + B_{21}D(A_{12})$.*

Proof. Compute

$$\begin{aligned}
&\Delta\big([A_{12}, B_{21}]\big) - D\big([A_{12}, B_{21}]\big)\\
&\quad = \big[\Delta(A_{12}), B_{21}\big] + \big[A_{12}, \Delta(B_{21})\big] - D(A_{12}B_{21} - B_{21}A_{12})\\
&\quad = \big[D(A_{12}), B_{21}\big] + \big[A_{12}, D(B_{21})\big] - D(A_{12}B_{21} - B_{21}A_{12})\\
&\quad = D(A_{12})B_{21} + A_{12}D(B_{21}) - D(A_{12}B_{21}) - B_{21}D(A_{12}) - D(B_{21})A_{12} + D(B_{21}A_{12}).
\end{aligned}$$

Since

$$\Delta\big([A_{12}, B_{21}]\big) - D\big([A_{12}, B_{21}]\big) = \Delta(A_{12}B_{21} - B_{21}A_{12}) - D(A_{12}B_{21} - B_{21}A_{12}),$$

by Lemma 2.15,

$$D(A_{12})B_{21} + A_{12}D(B_{21}) - D(A_{12}B_{21}) - B_{21}D(A_{12}) - D(B_{21})A_{12} + D(B_{21}A_{12}) \in \mathbb{F}I.$$

Hence there exists a scalar λ such that $D(A_{12}B_{21}) = D(A_{12})B_{21} + A_{12}D(B_{21}) + \lambda P_1$ and $D(B_{21}A_{12}) = B_{21}D(A_{12}) + D(B_{21})A_{12} + \lambda P_2$.

Now it is sufficient to show that $\lambda = 0$. For this, we distinguish two cases.

Case 1. The dimension of X is greater than 2. Then P_2 is of rank $\geqslant 2$. So there exists $C_{12} \in \mathscr{A}_{12}$ such that it is linearly independent of A_{12}. By the proof in the preceding paragraph, there exists a scalar μ such that $D(B_{21}C_{12}) = D(B_{21})C_{12} + B_{21}D(C_{12}) + \mu P_2$. Then by Lemma 2.11, we have

$$\begin{aligned}
D(A_{12}B_{21}C_{12}) &= D(A_{12}B_{21})C_{12} + A_{12}B_{21}D(C_{12})\\
&= \big(D(A_{12})B_{21} + A_{12}D(B_{21}) + \lambda P_1\big)C_{12} + A_{12}B_{21}D(C_{12}),
\end{aligned}$$

and

$$\begin{aligned}
D(A_{12}B_{21}C_{12}) &= D(A_{12})B_{21}C_{12} + A_{12}D(B_{21}C_{12})\\
&= D(A_{12})B_{21}C_{12} + A_{12}\big(D(B_{21})C_{12} + B_{21}D(C_{12}) + \mu P_2\big).
\end{aligned}$$

Comparing these two equations, we get $\lambda C_{12} = \mu A_{12}$, and hence $\lambda = \mu = 0$ by the linear independence of C_{12} and A_{12}.

Case 2. The dimension of X is equal to 2. Then we can identify $B(X)$ with $M_2(\mathbb{F})$, the algebra of two-by-two matrices over $\mathbb{F}$. Let E_{ij} be the matrix unit, $1 \leqslant i, j \leqslant 2$. Note that we have assumed that $E_{11} = P_1$ and $E_{22} = P_2$. Then $\mathscr{A}_{ij} = \mathbb{F}E_{ij}$, $1 \leqslant i, j \leqslant 2$. Note that $D(\mathscr{A}_{ij}) \subseteq \mathscr{A}_{ij}$, $1 \leqslant i, j \leqslant 2$. It follows, for $i \in \{1, 2\}$, that there is a function $g_i : \mathbb{F} \to \mathbb{F}$ such that

$$D(aE_{ii}) = g_i(a)E_{ii}$$

for all $a \in \mathbb{F}$. Then by Lemma 2.11, for $a \in \mathbb{F}$, we have

$$D(aE_{12}) = D\big((aE_{11})E_{12}\big) = g_1(a)E_{12} + aD(E_{12})$$

and

$$D(aE_{12}) = D\big(E_{12}(aE_{22})\big) = g_2(a)E_{12} + aD(E_{12}).$$

So $g_1(a) = g_2(a)$ for all $a \in \mathbb{F}$. Consequently, for all $a \in \mathbb{F}$,

$$\operatorname{tr}\big(D(aE_{11} - aE_{22})\big) = \operatorname{tr}\big(g_1(a)E_{11} - g_1(a)E_{22}\big) = 0.$$

In particular,

$$\operatorname{tr}(D(A_{12}B_{21} - B_{21}A_{12})) = 0.$$

On the other hand

$$\operatorname{tr}\big(\Delta\big([A_{12}, B_{21}]\big)\big) = \operatorname{tr}\big(\big[\Delta(A_{12}), B_{21}\big]\big) + \operatorname{tr}\big(\big[A_{12}, \Delta(B_{21})\big]\big) = 0.$$

Therefore,

$$\operatorname{tr}\big(\Delta\big([A_{12}, B_{21}]\big) - D\big([A_{12}, B_{21}]\big)\big) = 0.$$

This together with Lemma 2.15 gives that $\Delta([A_{12}, B_{21}]) - D([A_{12}, B_{21}]) = 0$. Hence $D(A_{12}B_{21}) = D(A_{12})B_{21} + A_{12}D(B_{21})$ and $D(B_{21}A_{12}) = D(B_{21})A_{12} + B_{21}D(A_{12})$. $\square$

Now, by Lemmas 2.11, 2.14 and 2.16, we see that D satisfies the derivation equation. To complete the proof of the theorem, for $A \in B(X)$, we define $\tau(A) = \Delta(A) - D(A)$. Thus, if $i = j$, then $\tau(A_{ij}) = f_i(A_{ij})I$; otherwise $\tau(A_{ij}) = 0$. To show $\tau(A) \in \mathbb{F}I$ for all $A \in B(X)$, we need two lemmas.

Lemma 2.17. *For all $A \in B(X)$, there holds $P_1\Delta(A)P_1 + P_2\Delta(A)P_2 - \Delta(P_1AP_1) - \Delta(P_2AP_2) \in \mathbb{F}I$.*

Proof. Let $T_{12} \in \mathscr{A}_{12}$ and $A \in B(X)$. Since

$$\big[P_1, [T_{12}, A]\big] = [P_1, T_{12}A - AT_{12}] = T_{12}P_2AP_2 - P_1AP_1T_{12},$$

it follows

$$\begin{aligned}\Delta(T_{12}P_2AP_2 - P_1AP_1T_{12}) &= \Delta\big(\big[P_1, [T_{12}, A]\big]\big) \\ &= \big[P_1, \big[\Delta(T_{12}), A\big]\big] + \big[P_1, \big[T_{12}, \Delta(A)\big]\big] \\ &= \Delta(T_{12})P_2AP_2 - P_1AP_1\Delta(T_{12}) + T_{12}P_2\Delta(A)P_2 - P_1\Delta(A)P_1T_{12}.\end{aligned}$$

On the other hand, by Lemma 2.8, we have

$$\begin{aligned}&\Delta(T_{12}P_2AP_2 - P_1AP_1T_{12}) \\ &\quad = \Delta(T_{12}P_2AP_2) - \Delta(P_1AP_1T_{12}) \\ &\quad = \Delta\big([T_{12}, P_2AP_2]\big) - \Delta\big([P_1AP_1, T_{12}]\big) \\ &\quad = \big[\Delta(T_{12}), P_2AP_2\big] + \big[T_{12}, \Delta(P_2AP_2)\big] - \big[\Delta(P_1AP_1), T_{12}\big] - \big[P_1AP_1, \Delta(T_{12})\big] \\ &\quad = \Delta(T_{12})P_2AP_2 + T_{12}\Delta(P_2AP_2) - \Delta(P_2AP_2)T_{12} - \Delta(P_1AP_1)T_{12} + T_{12}\Delta(P_1AP_1) - P_1AP_1\Delta(T_{12}).\end{aligned}$$

Comparing these two equations, we get

$$\big(P_1\Delta(A)P_1 - \Delta(P_1AP_1) - \Delta(P_2AP_2)\big)T_{12} = T_{12}\big(P_2\Delta(A)P_2 - \Delta(P_1AP_1) - \Delta(P_2AP_2)\big).$$

Hence for all $T_{12} \in \mathcal{A}_{12}$,

$$\begin{aligned}&\big(P_1\Delta(A)P_1 + P_2\Delta(A)P_2 - \Delta(P_1AP_1) - \Delta(P_2AP_2)\big)T_{12} \\ &\quad = T_{12}\big(P_1\Delta(A)P_1 + P_2\Delta(A)P_2 - \Delta(P_1AP_1) - \Delta(P_2AP_2)\big).\end{aligned}$$

This together with the fact

$$P_1\Delta(A)P_1 + P_2\Delta(A)P_2 - \Delta(P_1AP_1) - \Delta(P_2AP_2) \in \mathscr{A}_{11} + \mathscr{A}_{22}$$

gives the desired results. $\square$

Lemma 2.18. *For all $A \in B(X)$, there holds $\Delta(A) - \sum_{i,j=1}^{2} \Delta(P_iAP_j) \in \mathbb{F}I$.*

Proof. By Lemmas 2.4 and 2.8, we have

$$P_1\Delta(A)P_2 + P_2\Delta(A)P_1 = \Delta(P_1AP_2 + P_2AP_1) = \Delta(P_1AP_2) + \Delta(P_2AP_1).$$

So

$$\begin{aligned}\Delta(A) - \sum_{i,j=1}^{2} \Delta(P_iAP_j) &= P_1\Delta(A)P_1 + P_2\Delta(A)P_2 + P_1\Delta(A)P_2 + P_2\Delta(A)P_1 \\ &\quad - \Delta(P_1AP_1) - \Delta(P_2AP_2) - \Delta(P_1AP_2) - \Delta(P_2AP_1) \\ &= P_1\Delta(A)P_1 + P_2\Delta(A)P_2 - \Delta(P_1AP_1) - \Delta(P_2AP_2) \in \mathbb{F}I. \qquad \square\end{aligned}$$

Now by Lemma 2.18 and the definition of D and τ, we see that $\tau(A) \in \mathbb{F}I$ for all $A \in B(X)$. Hence, for $A, B \in B(X)$, since D is an additive Lie derivation, it follows that

$$\begin{aligned}\tau\big([A,B]\big) &= \Delta\big([A,B]\big) - D\big([A,B]\big)\\ &= \big[\Delta(A), B\big] + \big[A, \Delta(B)\big] - D\big([A,B]\big)\\ &= \big[D(A), B\big] + \big[A, D(B)\big] - D\big([A,B]\big) = 0.\end{aligned}$$

The proof is complete.

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